# Understanding the Space of $2 \times 2$ Games

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**Abstract**

The space of $2 \times 2$ games is four-dimensional and has a complex but understandable structure. In this paper, we describe and discuss two ways of assigning coordinates to $2 \times 2$ games. The $u$-$v$ system is simpler; it is efficient, effective, and is always continuous. The $x$-$b$ system is more readily understandable, more presentable, and eliminates redundancies present in the $u$-$v$ system. Using $x$-$b$ coordinates, we explore the space of $2 \times 2$ games through two-dimensional slices, and develop an understanding of how the parts of this space are connected. Finally, we use an interactive computer program to move through the space, allowing us to observe the intricacies of the space's structure.

## 1 Introduction

Game theory is a study of the behavior of logical actors. More specifically, it supposes that the decisions made by one individual can affect other individuals' results. There is no assumption of competition nor cooperation: we assume that each player seeks only to maximize their own return, and is not concerned with returns of other players. We call these systems of decisions and returns *games*. Understanding these games and the relationships between them provides insight into how people behave, how people should behave, and how changing people's incentives might affect their behavior.

In this paper, we will explore the space of $2 \times 2$ games. In Section 2, the concept of a game is introduced, and both types of equilibria are discussed: pure equilibria and mixed equilibria. In Section 3, we represent games as simple, intuitive diagrams. In Section 4, we establish a method of assigning coordinates to any game; this expands the work done by Roberts and Gunderson [2]. In Section 5, we build a new system of coordinates which is more easily interpreted, and eliminates redundancies present in the previous system; this system is analogous to a discrete categorization of games presented by Robinson and Goforth [1], and Bruns [4], but here it is made continuous. In Section 6, we discuss how our new coordinates are valuable using a key figure. In Section 7, an interactive version of that figure is described.

## 2 Games and Equilibria

In a $2\times2$ *game*, two players simultaneously choose between two options. Each player then receives a *return*, which depends on both the choice they made and the choice made by the other player. Thus, there are four possibilities:

- each player chooses their first option,
- each player chooses their second option,
- the first player chooses their first option and the second player chooses their second option,
- or the first player chooses their second option and the second player chooses their first option.

Each player might receive a different return in each of these cases, so any game involves exactly eight return values. These values are typically given in two $2 \times 2$ *payoff matrices*: one per player. One player chooses between the rows, and the other chooses between the columns. We call these players the *row player* and the *column player* respectively. It is important to note that each player knows the contents of both players' matrices; we will always assume that this is the case.

Suppose the row player has a payoff matrix $R$ and the column player has a payoff matrix $C$ given by

$$R = \begin{pmatrix} \mathbf{6} & \underline{\mathbf{0}} \\ 2 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 4 & \underline{\mathbf{6}} \\ 6 & \mathbf{3} \end{pmatrix}. \tag{1}$$

If the row player chooses the first row and the column player chooses the second column, then the row player receives a return of 0, and the column player receives a return of 6.

## 2.1 Pure Equilibria

A *pure strategy* is one in which a player will always choose the same option. This greatly clarifies the other player's best choice if they want to optimize their returns. Suppose the row player knows that the column player is choosing the first column, so they choose the row with the greatest entry in that column to maximize their return. If neither player can change their choice to receive a greater return, then the choices being made are a *pure equilibrium.*

*Pure equilibria* are common, existing in 7/8 of games, as we will see later, though none exist in Game (1). In Game (2), given by

$$R = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \text{ and } C = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \tag{2}$$

there are two pure equilibria: each player can choose their first option, or each player can choose their second option.

## 2.2 Mixed Equilibria

A *mixed strategy* is one in which a player makes one choice with some probability $p$ and the other with probability $1 - p$. If both players are playing mixed strategies and neither player can change their strategy to increase their average (or *expected*) return, then those strategies are a *mixed equilibrium.* Games typically have either zero or one mixed equilibria, but some edge cases have infinitely many; Game (2) is an example of such a game. Mixed equilibria exist in only 1/4 of games.

We will denote a mixed equilibrium $(p, q)$, where $p$ is the probability that the row player chooses the first row and $q$ is the probability that the column player chooses the first column. Suppose a mixed equilibrium exists for some game. When players play at that equilibrium, a player's expected return given that they choose their first option equals the expected return given that they choose their second option. If the row and column players have payoff matrices given by

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } C = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix},$$

then

$$q \cdot a + (1 - q) \cdot b = q \cdot c + (1 - q) \cdot d$$

and

$$p \cdot \bar{a} + (1 - p) \cdot \bar{c} = p \cdot \bar{b} + (1 - p) \cdot \bar{d}.$$

This must be true because there cannot be a strategy that the column player, for example, can switch to which would increase their expected return. Using this fact and simple algebra, we get the following theorem.

**Theorem 1.** *Suppose a $2 \times 2$ game has a mixed equilibrium $(p, q)$ and is given by the row and column players' payoff matrices*

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } C = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

$$\text{Then } p = \frac{\bar{d} - \bar{c}}{\bar{a} - \bar{b} - \bar{c} + \bar{d}} \text{ and } q = \frac{d - b}{a - b - c + d}.$$

*Furthermore, the expected returns of the row and column players are given by*

$$E_R = \frac{ad - bc}{a - b - c + d} \text{ and } E_C = \frac{\bar{a}\bar{d} - \bar{b}\bar{c}}{\bar{a} - \bar{b} - \bar{c} + \bar{d}}$$

*respectively, assuming that players are playing at the mixed equilibrium.*

Observe that each player's expected return depends on that player's payoff matrix, as might be expected, but players' strategies depend only on the other player's matrix.

# 3 Game Diagrams

Payoff matrices can be hard to interpret at a glance, and we are going to see a lot of games, so we want to devise a more comprehensible way to illustrate a game. Figure 1 is such an illustration. For payoff matrices

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } C = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix},$$

belonging to the row and column players, we consider the points $(a, \bar{a})$, $(b, \bar{b})$, $(c, \bar{c})$, and $(d, \bar{d})$. We connect two of these points by a cerulean line if they are

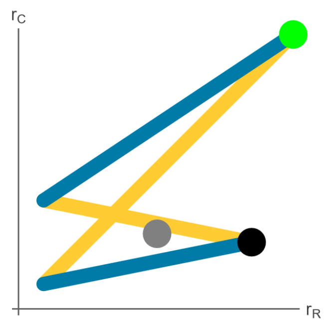


Figure 1: The diagram for the game given by the payoff matrices $R = \begin{pmatrix} 0 & 6 \\ 5 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 6 \\ 1 & 0 \end{pmatrix}$. The horizontal coordinate is the row player's return and the vertical coordinate is the column player's return.

taken from the same row, and connect two points by a gold line if they are taken from the same column. We draw points for any pure equilibria at the corresponding vertices of the quadrilateral. These points are colored based on the following criteria: if each player gets the best return in their matrix, the equilibrium is light green; if only the row player gets the best return in their matrix, the point is gold; if only the column player gets their best return, the point is cerulean; and if neither player gets their best return, the equilibrium is black. For a mixed equilibrium, we draw a gray point at $(r_a, r_b)$, where $r_a$ is the expected return for the row player and $r_b$ is the expected return for the column player when playing at the mixed equilibrium.

The row player wants to "move" to the right, but can only move along the gold lines (by switching to the other row). The column player can "move" along the cerulean lines by switching to the other column, and wants to move upward. We associate each player with the color of line they can move along: this is why the pure equilibria are sometimes colored gold or cerulean.

# 4 A Global Coordinate System

While a game has eight variables, we can eliminate two degrees of freedom per matrix without meaningfully changing the game. Multiplying either matrix by a positive scalar or adding a scalar to either matrix does not affect the way the game is played; it only affects the players' returns. We normalize matrices to have a minimum entry of zero and a maximum entry of six; we are ignoring the case where all entries are equal. This leaves us with four remaining variables. This four-dimensional space is topologically equivalent to the product of two spheres: one sphere per matrix. Figure 2 shows the positions of the most degenerate matrices on one of these spheres. This sphere is best characterized as a cube, as we care about the triangular regions shown in Figure 2.

Each sphere has 24 triangular regions whose vertices correspond to matrices containing only zeroes and sixes. Two points in the same triangle correspond to matrices whose entries are ordered in the same way: the greatest entry is in the same position, the second greatest entry is in the same position, et cetera. As there are $4! = 24$ ways to order 4 entries, any nondegenerate matrix corresponds to a point in exactly one region. Note that the rectangles in top and bottom portions of Figure 2 are the triangles on the top and bottom sides of the cube, but have been stretched to fill the space. Every point with a $v$-coordinate of 2 maps to the same matrix, and the same is true for points with $v$-coordinate $-2$.

**Definition.** For a game given by

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } C = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix},$$

we define the coordinates $v_1$ and $u_1$ as follows:

$$v_1 = \frac{a}{6} - \frac{b}{6} + \frac{c}{6} - \frac{d}{6} \tag{3}$$

$$u_1 = \begin{cases} -\left(\frac{a}{6} - \frac{b}{6} - \frac{c}{6} + \frac{d}{6}\right) - 3 & \begin{array}{r}\text{if } (a = 6 \text{ or } b = 6) \\ \text{and } (c = 0 \text{ or } d = 0)\end{array} \\ -\left(\frac{a}{6} + \frac{b}{6} - \frac{c}{6} - \frac{d}{6}\right) - 1 & \begin{array}{r}\text{if } (b = 6 \text{ or } c = 6) \\ \text{and } (a = 0 \text{ or } d = 0)\end{array} \\ \frac{a}{6} - \frac{b}{6} - \frac{c}{6} + \frac{d}{6} + 1 & \begin{array}{r}\text{if } (a = 0 \text{ or } b = 0) \\ \text{and } (c = 6 \text{ or } d = 6)\end{array} \\ \frac{a}{6} + \frac{b}{6} - \frac{c}{6} - \frac{d}{6} + 3 & \begin{array}{r}\text{if } (b = 0 \text{ or } c = 0) \\ \text{and } (a = 6 \text{ or } d = 6)\end{array} \end{cases}. \tag{4}$$

We define $v_2$ and $u_2$ by applying precisely the same formulae to the transpose of the column player's matrix. So $v_2$ is given by

$$v_2 = \frac{\bar{a}}{6} - \frac{\bar{c}}{6} + \frac{\bar{b}}{6} - \frac{\bar{d}}{6}. \tag{5}$$

Equation (3) gives the latitude of the row player's matrix, which we call $v_1$, and ranges from $-2$ to 2 as we imagine the cube to have a side length of 2. Equation (4) gives the longitude of the row player's matrix, which we call $u_1$, and ranges from $-4$ to 4. Expressing $u_1$ is a more difficult task as the expression depends on which side of the cube a point is on. The latitude of the column player's matrix is called $v_2$, given by Equation (5), and the longitude is $u_2$.

When we move diagonally along one side of the cube, only one entry in our matrix changes, and when we move orthogonally across the side of a cube, two entries change together. When we move along or beside a red line shown in Figure 2, the second-greatest entry changes; when we move beside a blue line, the third-greatest entry changes; and when we move beside a green line, the second- and third-greatest entries both change. Furthermore, when we cross a red line, the two smallest entries are equal; when we cross a blue line, the two greatest entries are equal; and when we cross a green line, the second- and third-greatest entries are equal, and we pass to another side of the cube.

# 5 Another Coordinate System

We have established the $u_1$-$v_1$-$u_2$-$v_2$ coordinate system (the global coordinate system) as a meaningful way to describe a game. However, it is difficult to interpret the meanings of each coordinate or say anything substantive about a game after looking only at its global coordinates. Because the space is four-dimensional, we struggle to produce valuable visualizations for the entire set. Additionally, although any two functionally identical matrices will receive the same $u$-$v$ coordinates, two functionally identical games might have different $u_1$-$v_1$-$u_2$-$v_2$ coordinates. Switching the rows of one player's matrix meaningfully alters the game, but switching the rows or columns of both players' matrices does not. Thus, for any nondegenerate game, there are three other games which are exactly the same in practice, despite having

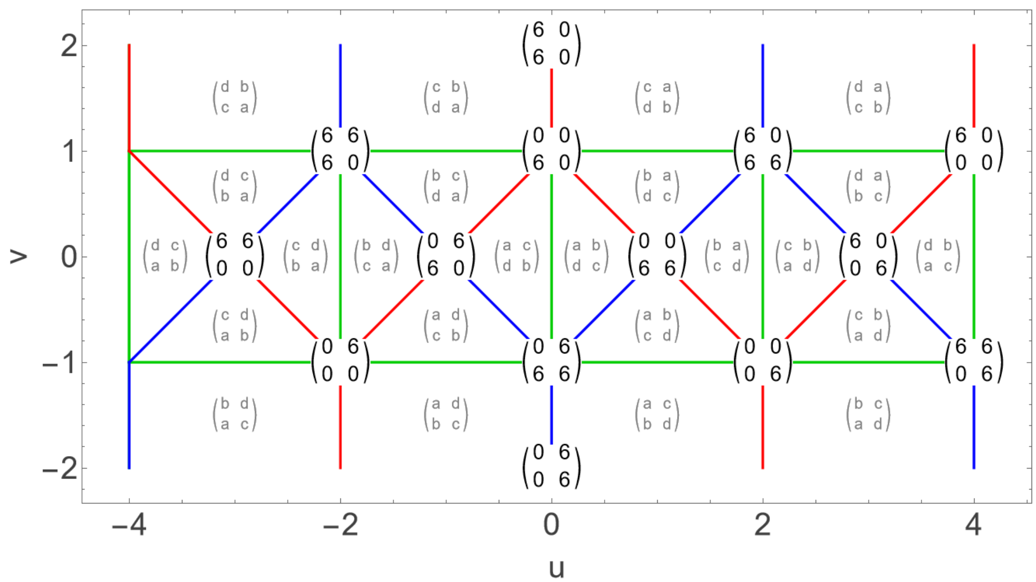


Figure 2: The vertical coordinate $v$ is given by (3) and the horizontal coordinate $u$ is given by (4), modified by stretching the triangles on the top and bottom of the picture. For all of the gray matrices, we assume $a < b < c < d$.

different coordinates. Note that all four of these games would produce the same diagram. Recall that each sphere is broken into 24 regions. Those regions become $24^2 = 576$ regions when both spheres are involved, but this fourfold redundancy reduces that number to $24^2/4 = 144$ distinct regions.

We define $b_1$ as the difference between the two greatest entries of the row player's payoff matrix. In Figure 3, the points corresponding to games such that $b_1 = 2$ are colored blue. Observe that there are four distinct triangles (one of the sides has been distorted on each). Switching the rows or columns of a matrix from one triangle will result in a matrix from another triangle. In this way, we can treat the triangles as one and the same. To illustrate this, consider the matrices

$$A_1 = \begin{pmatrix} 0 & 6 \\ 4 & 3 \end{pmatrix},\ A_2 = \begin{pmatrix} 4 & 3 \\ 0 & 6 \end{pmatrix},\ A_3 = \begin{pmatrix} 6 & 0 \\ 3 & 4 \end{pmatrix} \text{ and, } A_4 = \begin{pmatrix} 3 & 4 \\ 6 & 0 \end{pmatrix}. \tag{6}$$

The points which these four matrices map to are shown in Figure 3.

We define $x_1$ as the distance traveled around a triangle, ranging from zero at the start, to six when a full lap has been completed. We let $x_1 = 0 = 6$

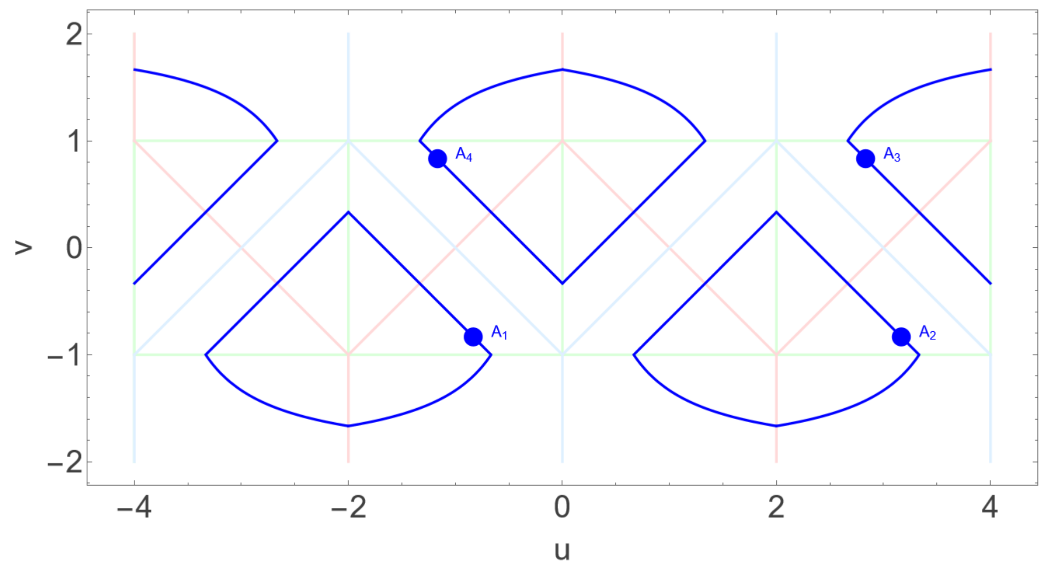


Figure 3: Each point on a solid blue line maps to a matrix whose two greatest entries differ by 2. The four blue points map to the matrices in Equation (6).

at the most extreme point of a triangle (the highest point if the triangle is in the top or the lowest point if the triangle is in the bottom), and we let $x_1$ increase as we move along a triangle in a clockwise direction. We define $b_2$ and $x_2$ for the other player in the same fashion.

Thus, we have a new coordinate system (a local coordinate system), but for any coordinates $(x_1, b_1, x_2, b_2)$, each player's matrix could be any of four possibilities, giving 16 possible combinations. However, for any one of those possibilities, three of the other possibilities are functionally identical, so there are only four distinct games associated with given coordinates $(x_1, b_1, x_2, b_2)$. If we fix $b_1$ and $b_2$, we get four different products of two triangles, which pass through each of the $6 \cdot 6 \cdot 4 = 144$ regions of the four-dimensional space exactly once. These four products are depicted in Figure 4, which is discussed in the next section.

# 6 A More Insightful Geography

To create Figure 4, we suppose $b_1$ and $b_2$ both equal 2. Red lines are drawn where a triangle crosses a red line on its cube, and green lines are drawn where

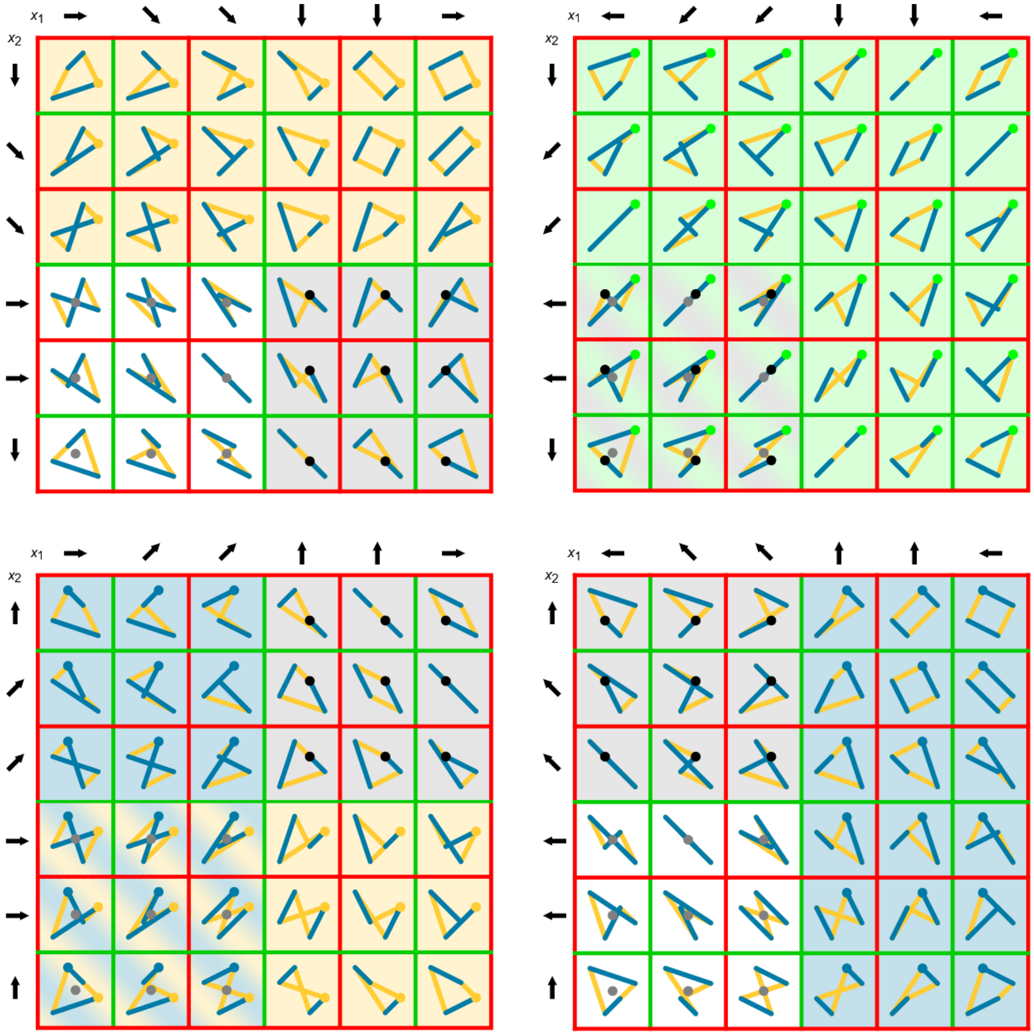


Figure 4: Gives examples of all 144 classes of games. These represents the 4 tori produced by supposing $b_1 = b_2 = 2$.

a triangle crosses a green line. As previously described, when a game lies on a red boundary, the two smallest entries in one player's matrix are equal, and when a game lies on a green boundary, a player's second- and third-greatest entries are equal. So degenerate games lie on these lines, and games on the intersections of these lines are doubly degenerate. These boundaries separate the 144 regions, and the diagram of a game from each region is shown in the appropriate cell.

Each quadrant allows for wrapping, which is to say that the top and bottom rows are adjacent and the left and right columns are adjacent. Some regions are adjacent, but not obviously so. When either $b_1$ or $b_2$ is zero, the triangles in Figure 3 overlap and there are regions containing identical games; two regions whose games become identical in this way are adjacent, sharing a blue boundary. The arrows along the sides of each quadrant indicate that each region in that row or column shares a blue boundary with a region in the quadrant the arrow points to; the region in question is in nearly the same position within its quadrant, only flipped over the nearest red line. The arrows come in three classes—horizontal, vertical, and diagonal—and the positions of the arrows in each of these classes are consistent between quadrants. Observe that a region on the edge of a quadrant shares a blue boundary with the region directly across from it on the opposite quadrant.

When $b_1$ or $b_2$ is zero, each game is identical to a game in another quadrant. When both $b_1$ and $b_2$ are zero, crossing either blue boundary takes us to an identical game, and crossing both boundaries takes us to a fourth identical game. Typically, these four games appear in four different quadrants, but in the case where the blue boundaries are shared with the same quadrant, crossing both of them takes us back to the quadrant where we began. In this way, when $b_1$ and $b_2$ are zero, certain regions within each quadrant are identical. There is yet more complexity: in the case where a game is on a green boundary and a blue boundary of the same player, the two closest red lines are equidistant, so an identical game can be reached by flipping over either red line and moving into the appropriate quadrant. Thus, it may be that a game appears exactly three times within our space. This becomes apparent when we look at Figure 3; when the triangles overlap, the matrices on green lines are in three different triangles simultaneously. If a game is on two blue boundaries and two green boundaries (so both players' matrices have three sixes), then that game appears nine times within our space.

The quadrants are characterized by the positions of the greatest entries in players' matrices; with our normalizations, these entries are sixes. In the top-right quadrant, two sixes are in the same position, meaning players prefer the same scenario. We call this the good quadrant. In the bottom-right quadrant, there is a six in the column player's matrix which lies in the same row (but not in the same column) as a six in the row player's matrix. This is desirable for the column player, because the row player is likely to favor the row containing the column player's six. We call this the column player's quadrant. In the top-left quadrant, there is a six in the column player's matrix which lies in the same column (but not in the same row) as a six in the row player's matrix. This is desirable for the row player, because the column player is likely to favor the column containing the row player's six. We call this the row player's quadrant. Finally, in the bottom-left quadrant, there is a six in the column player's matrix which lies on the same diagonal as a six in the row player's matrix; in other words, these sixes are in neither the same row nor the same column. We call this the bad quadrant.

Notice that the descriptions of the games within each quadrant are not mutually exclusive if a matrix has multiple sixes. A player's matrix has multiple sixes if and only if that player's $b$ equals 0. This aligns with the behavior of the blue boundaries as described previously.

Games in the regions colored light green have pure equilibria at which both players get their best returns. (These equilibria are colored green). We call the 27 cells colored only light green "the good ell". We color cells where games have gold equilibria a light gold color, and the row player's ell is gold and wraps from the top of the row player's quadrant to the bottom of the bad quadrant. Cells whose games have cerulean equilibria or black equilibria are colored light cerulean or gray respectively. The column player's ell is cerulean and wraps from the right of the column player's quadrant to the left of the bad quadrant. The bad ell is gray and lies in three quadrants: the bad quadrant, the row player's quadrant, and the column player's quadrant. The infamous prisoner's dilemma is at the crook of this bad ell, in the top right of the bad quadrant. Mixed equilibria occur exactly in the bottom left quarter of each quadrant.

# 7 An Interactive Picture

While Figure 4 is very insightful, it is just a slice of the four-dimensional space. To explore the whole space, we must allow $b_1$ and $b_2$ to change. However, when $b_1$ is larger, the triangle becomes smaller, and $x_1$ has less

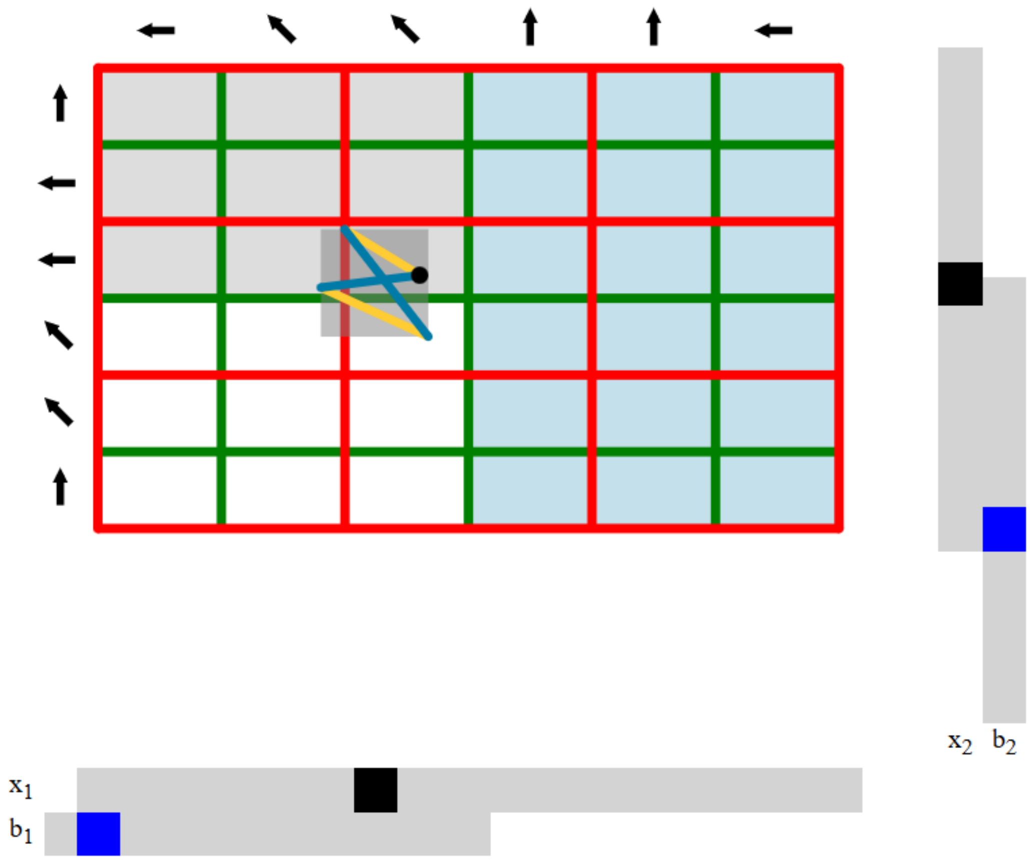


Figure 5: A screenshot of the web app. We can change the values of $x_1$, $b_1$, $x_2$, and $b_2$ as we wish, and the diagram is centered on the point $(x_1, x_2)$. The picture becomes squished when $b_1$ and $b_2$ become bigger.

impact on the game. At the extreme, when $b_1 = 6$, changing $x_1$ has no impact on the game. The same is true for $b_2$ and $x_2$. To reflect this, when $b_1$ becomes larger, the four quadrants in Figure 4 should be squished horizontally so that when $b_1 = 6$, they each have zero width. This clearly indicates how changing $b_1$ affects the impact $x_1$. When $b_2$ becomes larger, the quadrants should be squished vertically to reflect the impact of $x_2$.

If our picture is interactive and changing as we manipulate $b_1$ and $b_2$, we don't need to show all four parts of Figure 4. Suppose we only show one of the four. Additionally, suppose that rather than showing 36 diagrams, we show only one, and we allow ourselves the luxury of manipulating $x_1$ and $x_2$. Now we sit at a point in the four-dimensional space, with the ability to manipulate any of the four variables. When $b_1$ or $b_2$ becomes zero, the game becomes identical to the game at another point on another quadrant. Thus, it's only natural when we change $b_1$ or $b_2$ to zero, that we might jump to another quadrant. This is the application I created using JavaScript. Figure 5 is a screenshot of that application, which is accessible at https://elliothanson.com/game-theory.

There are multiple ways to navigate our space using the app. First, you can use the sliders to manipulate $x_1$, $x_2$, $b_1$, and $b_2$. When moving $b_1$ or $b_2$ onto or off of zero will trigger the program to switch quadrants, the appropriate arrows turn blue (the ones on the top if $b_1$ is changing and the ones on the side if $b_2$ is changing). Additionally, when either $b_1$ or $b_2$ is zero, a button appears, allowing the user to easily cross the blue boundary.

Clicking or dragging your mouse over the figure allows you to manipulate $x_1$ and $x_2$; clicking (and not dragging) uses less precision, making it easier to place the diagram directly on a boundary or in the center of a cell.

Finally, the keyboard can be used to change the coordinate values. The arrow keys can be used to change $x_1$ and $x_2$. The 'a' and 'd' keys can be used to change $b_1$ and the 'w' and 's' keys change $b_2$. While the 'Shift' key is being held, using the keys to change the game will instead add velocities to the corresponding parameters, allowing us to move through the space along a set trajectory. Pressing the space bar sets all velocities to zero. When $b_1$ or $b_2$ becomes zero as the result of a key press or while the parameters move on their own, the program immediately switches to the appropriate quadrant.

# 8 Conclusion

The space of $2 \times 2$ games is multifaceted and interconnected in a complex way. The literature addresses some of this structure; the 144 regions and the connections between them are precedented: Figure 4 closely resembles figures made by Robinson and Goforth [1]. However, this space has historically been thought of as a discrete set of 144 games. By viewing the space of games as a continuum, we allow for the variation which can occur in each region: the players' behavior will be fundamentally the same, but their returns may vary. Additionally, the system we have established does not exclude degenerate games. The nuances of this space come largely from the behavior of games on the boundaries. These games had been largely ignored. This paper provides a window into the whole space of $2 \times 2$ so that we may see and understand it in its entirety.

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